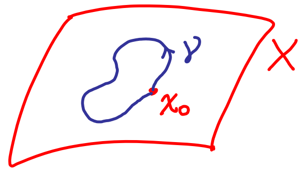
Algebraic Topology by Greenberg

Note by Conan

(I) Elementary Homotopy Theory

§ Fundamental group

$$\pi_1(X, x_0) = [(S^1, 1), (X, x_0)]$$

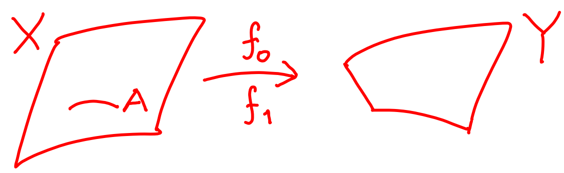


homotopy classes of loops in X at x_0.

- Group (multi. = composition)
- X path conn. $\Longrightarrow$ $\pi_1(X, x_0)$ indep. of x_0, up to isom.
- $f : (X, x_0) \longrightarrow (Y, y_0) \rightsquigarrow f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$

Def. $f_0, f_1 : X \longrightarrow Y$

$f_0|_A = f_1|_A$



$f_0 \sim f_1$ rel A homotopy

$\overset{\Delta}{\Longleftrightarrow}$ $\exists\, F : X \times [0, 1] \longrightarrow Y$

$F|_{X \times 0} = f_0$, $F|_{X \times 1} = f_1$, $F|_{A \times I}$ indep. of I

Def. $X \sim Y$ homotopy equivalent

$\overset{\Delta}{\Longleftrightarrow}$ $\exists\, X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t. $g \circ f \sim 1_X$ & $f \circ g \sim 1_Y$

Def. $X \sim$ pt. contractible ($\Rightarrow \pi_1(X) = 0$)

- $f \sim g \Longrightarrow f_* \cong g_*$ on π_1
- $X \sim Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$

Theorem $\pi_1(S^1) \cong \mathbb{Z}$

Note: $S^1 = \mathbb{R}/\mathbb{Z}$

In general, $\underset{\text{discrete}}{\Gamma} \triangleleft \underset{\pi_1=0,\ \text{Lie gp.}}{G} \Rightarrow \pi_1(G/\Gamma) \simeq \Gamma$

(Pf: $\pi_1(G/\Gamma, e) \longrightarrow \Gamma$
$\gamma \longmapsto \tilde{\gamma}(1)$ path lifting

- Application: $S^1 = \partial D^2$ is not a retract of D^2

(Recall: $X \underset{r}{\overset{i}{\rightleftarrows}} Y$, r retract $\overset{\Delta}{\Leftrightarrow}$ $r \circ i = 1_X$)

Pf: IF $\exists$ retract

$$\begin{tikzcd} D^2 \arrow[r, "r"] & S^1 \\ S^1 \arrow[u, hook, "i"] \arrow[ur, equal, "1"'] \arrow[r, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} \underbrace{\pi_1(S^1)}_{\mathbb{Z}} \arrow[r, "i_*"] \arrow[rr, bend left, "1_* = id."] & \underbrace{\pi_1(D^2)}_{0} \arrow[r] & \underbrace{\pi_1(S^1)}_{\mathbb{Z}} \end{tikzcd} \quad (\rightarrow\!\leftarrow)$$

Cor: $f: D^2 \circlearrowleft \Rightarrow \exists$ fix point.

- $\pi_1(X \times Y) \simeq \pi_1(X) \times \pi_1(Y)$

eg. $\pi_1(T^2) = \mathbb{Z}^2$.

- $X = U \cup V$ w/ $\pi_1(U) = 0 = \pi_1(V)$ & $U \cap V$ path conn.

$\Longrightarrow \pi_1(X) = 0$

eg. $\pi_1(S^n) = 0$ if $n \geq 2$ $\quad (\because S^n = D^n_+ \underset{S^{n-1}}{\cup} D^n_-)$

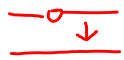
Remark: In general, $\pi_1(U \cup V) = \overset{\text{amalgamated sum}}{\pi_1(U) * \pi_1(V)}$,
called Van Kampen's theorem.

§ Covering spaces $E \xrightarrow{p} X$

$\Big(\triangleq \ \forall\, x \in X \ \exists$ nbd U s.t. $p^{-1}(U) = \coprod S_i$

$S_i \overset{\text{open}}{\subseteq} E$ s.t. $S_i \xrightarrow[\text{homeo.}]{p} U \Big)$

NOT: $\forall\, e \in E$ p is loc. homeo. near e.

counter-eg: 

e.g. $\mathbb{R} \longrightarrow S^1$

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, dashed, "\overset{?}{\exists} f'"] \arrow[r, "f"'] & (X, x_0) \end{tikzcd}$$

connected

<u>Unique Lifting Theorem.</u>

If $\exists\, f' \implies f'$ unique.

<u>Path Lifting Theorem.</u>

If $(Y, y_0) = ([0,1], 0) \implies \exists\, f'$

<u>Covering Homotopy Theorem</u>

If $F : Y \times I \to X$, $F|_{Y \times 0} = f$ $\implies$ $\exists\, F' : Y \times I \to F$, $F'|_{Y \times 0} = f'$

Assume $\exists\, f'$, $p \circ f' = f$ $\qquad$ $p \circ F' = F$

Cor: $p_* : \pi_1(E) \xhookrightarrow{1-1} \pi_1(X)$

$\pi_1(X) / p_* \pi_1(E) \overset{\text{bij.}}{=\!=} p^{-1}(x_0)$ if E conn.

(via lifting loops in X to path in E)

<u>Theorem</u>: If $\pi_1(E) = 0$, then

$$\pi_1(X) \simeq \left\{ \phi \ \middle| \ \begin{tikzcd} E \arrow[r, "\phi"', "\text{homeo.}"] \arrow[d] & E \arrow[d] \\ X \arrow[r, equal] & X \end{tikzcd} \right\}$$ group of covering transf.

§ Lifting criterion

(every space conn.)

$$\begin{tikzcd} & (E, e_0) \arrow[d, "p"] \arrow[d, phantom, shift left=6ex, "\text{covering}"] \\ (Y, y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\exists ?\, f'"] \arrow[r, phantom, shift left=4ex, "\circlearrowright"] & (X, x_0) \end{tikzcd}$$

<u>Theorem</u> $\exists f' \iff f_* \pi_1(Y) \subset p_* \pi_1(E)$.

In particular, covering $\tilde{X} \to X$ w/ $\pi_1(\tilde{X}) = 0$ is unique, called universal covering.

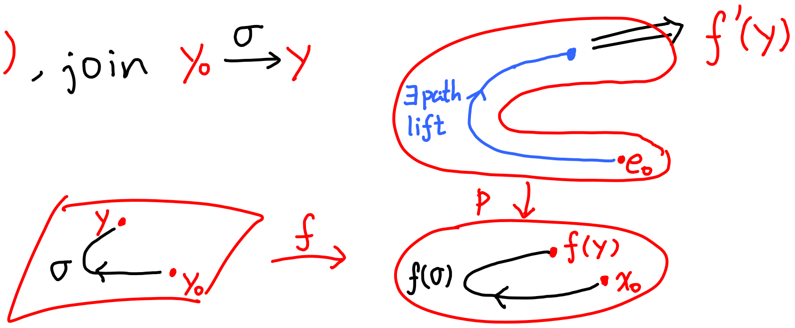
All other coverings $E \to X$ are below it, i.e.

$\tilde{X} \longrightarrow E \longrightarrow X$ (both are coverings)

(Remark: If small loops in X are contractible, (eg mfd.) then $\exists\ \tilde{X}$.)

Proof of thm. [$\Rightarrow$] $\because\ f_* = p_* \circ f'_*$

[$\Leftarrow$] Constr. $f'(y)$, join $y_0 \xrightarrow{\sigma} y$



Vary σ cts $\Rightarrow$ same $f'(y)$.

Choose different σ $\Rightarrow$ need π_1-condit^{n} to show same $f'(y)$.

QED.

$\circ = \bigcirc$

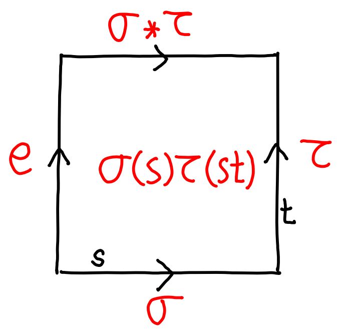
Prop. $\pi_1(G)$ Abelian $\forall\, G$ Lie group

Pf. $\sigma, \tau : I \longrightarrow G$ w/ $\sigma|_{\partial I} = \tau|_{\partial I} = e$

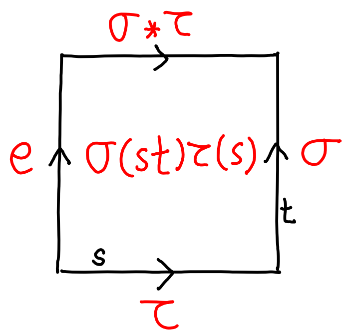
define $\sigma * \tau : I \longrightarrow G$ w/ $\sigma * \tau\,|_{\partial I} = e$

by $\sigma * \tau\,(t) := \sigma(t)\,\tau(t)$

$\sigma * \tau \sim \sigma\tau$



$\sigma * \tau \sim \tau\sigma$



§ Higher homotopy groups

based loop space $\qquad\qquad$ free loop space

$$\begin{tikzcd} \overbrace{\mathrm{Map}(S^1,X)_*}^{\Omega_{x_0}X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \overbrace{\mathrm{Map}(S^1,X)}^{\mathcal{L}X} \arrow[d] & \gamma \arrow[d, mapsto] \\ x_0 \arrow[r, phantom, "\in"] & X & \gamma(1) \end{tikzcd}$$

$$\Omega_{x_0}X := \mathrm{Map}(S^1, X)_* \leftarrow \text{based maps}$$
$$= \mathrm{Map}((I, \partial I), (X, x_0))$$

$$\pi_1(X, x_0) = [S^1, X]_*$$
$$= \pi_0(\Omega_{x_0}X) \text{ set of path conn. components.}$$

Def. $\pi_n(X, x_0) := \pi_{n-1}(\Omega_{x_0}X, \underset{\text{const. loop}}{x_0})$

Theorem $\pi_{\geq 2}(X)$ commutative.

[Pf $\Omega_{x_0}X$ 'likes' a group $\Rightarrow$ π_1 Abelian.

Theorem $E \xrightarrow{\text{covering}} X \Longrightarrow \pi_{\geq 2}(E) \xrightarrow{\cong} \pi_{\geq 2}(X)$

[Pf. Lifting criterion $\Longrightarrow$ $\exists!\, f'$

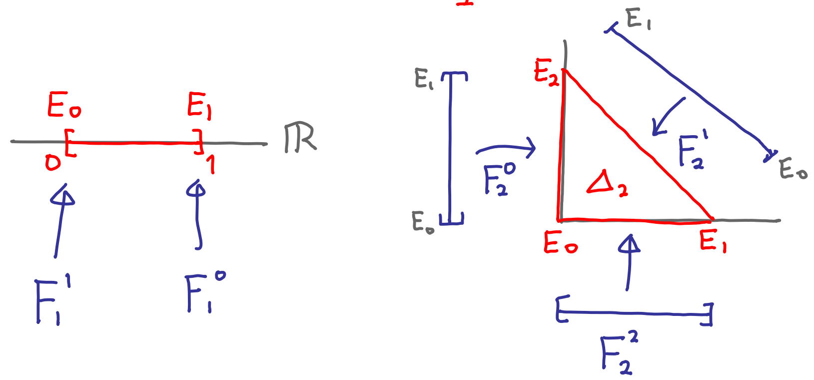
$$\begin{tikzcd} & & E \arrow[d, "p"', "\text{covering}"] \\ \pi_{n\geq 2}(X) \ni f : \underbrace{S^n}_{\pi_1=0} \arrow[rr] \arrow[urr, dashed, "f'"] & & X \end{tikzcd}$$

(II) Singular Homology

X any topo. space

$$S_q(X) := \mathbb{Z}\langle \underset{\text{singular chain}}{\sigma : \Delta_q \longrightarrow X} \rangle$$

$\Delta_1 =$



where $F_q^i : \Delta_{q-1} \longrightarrow \Delta_q$: affine map to $\partial\Delta_q$, omitting i^{th}-pt.

$\rightsquigarrow \partial : S_q(X) \longrightarrow S_{q-1}(X)$ " $\begin{tikzcd} \Delta_q \arrow[r, "c"] & X \\ \partial\Delta_q \arrow[u, phantom, "\cup"] \arrow[ur, "\partial c"'] & \end{tikzcd}$ "

$\partial^2 = 0$

Def: $\underset{\text{homology}}{H_q(X)} := \left.\dfrac{\mathrm{Ker}\,\partial}{\mathrm{Im}\,\partial}\right|_{S_q(X)} = \dfrac{Z_q}{B_q}$ $\begin{matrix}\text{cycles}\\ \text{boundaries}\end{matrix}$

(Can replace coeff. $\mathbb{Z}$ by any comm. ring R).

- $H_0(X) \simeq \mathbb{Z}^{b_0}$ w/ $b_0 = \#$ path comp. of X

- Naturality: $f : X \longrightarrow Y$

  $\rightsquigarrow f_* : (S_*(X), \partial) \longrightarrow (S_*(Y), \partial)$

  $\rightsquigarrow f_* : H_*(X) \longrightarrow H_*(Y)$.

§ Chain Complex

- $C = \{ \cdots C_{q+1} \xrightarrow{\partial} C_q \xrightarrow{\partial} C_{q-1} \to \cdots , \partial^2 = 0 \}$ as R-mod

$\rightsquigarrow H_*(C) \triangleq \dfrac{\operatorname{Ker}\partial}{\operatorname{Im}\partial} = \dfrac{Z_*}{B_*}$

- (C_*, ∂) exact $\iff H_*(C) = 0$

- $f : C \to C'$ chain map
$$\begin{tikzcd} \cdots C \arrow[r, "\partial"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowright"] & C \cdots \arrow[d] \\ \cdots C' \arrow[r, "\partial"'] & C' \cdots \end{tikzcd}$$

$\rightsquigarrow f_* : H_*(C) \to H_*(C')$

Prop. $f \simeq g : C \to C'$ chain homotopy

(i.e. $f - g = \partial' D + D\partial \quad \exists D : C \to C'[+1]$
$=: \{\partial, D\}$)

$\Rightarrow f_* = g_* : H_*(C) \to H_*(C')$.

Theorem. $\pi_*(X) = 0 \implies H_{*\neq 0}(X) = 0$
i.e. aspherical $\qquad S(X)$ acyclic.

Pf: $C_* := S_*(X) \xrightarrow{\varepsilon} \mathbb{Z}$ $(\simeq H_*(pt))$ augmentation

$C_0 \ni \Sigma a_i x_i \longmapsto \Sigma a_i$

Pick $b \in X \rightsquigarrow$ splitting $C_* \xleftarrow{\eta} \mathbb{Z}$ w/ $\varepsilon \circ \eta = 1_{\mathbb{Z}}$
$\eta(1) = b$

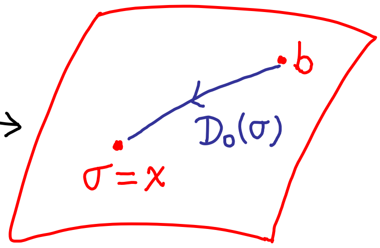
Want $\eta \circ \varepsilon \simeq 1_{C_*}$ ($\Rightarrow C_* \simeq \mathbb{Z}$ acyclic)

i.e. $\eta \circ \varepsilon - 1_{C_*} = \partial D + D\partial$

$\exists D_q : S_q(X) \longrightarrow S_{q+1}(X)$

$D_0 : S_0(X) \longrightarrow S_1(X)$

defined as $\rightsquigarrow$

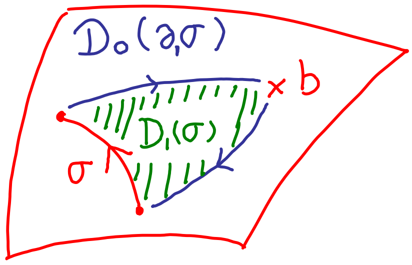


(okay $\because$ X connected)

$$\partial_1 D_0(\sigma) + \cancel{D_{-1}}\partial_0 = b - x = \eta \underbrace{\varepsilon(\sigma)}_{1} - \sigma$$

$D_1 : S_1(X) \longrightarrow S_2(X)$

$\exists D_1(\sigma)$ $(\because \pi_1(X) = 0)$



$$\partial_2 D_1(\sigma) + D_0(\partial_1 \sigma)$$
$$= (\sigma - D_0(\partial_1\sigma)) + D_0(\partial_1\sigma) = \sigma \overset{(\because\, \varepsilon(\sigma)=0)}{=} (1 - \eta\,\varepsilon)\sigma$$

Inductively $\rightsquigarrow$ D w/ $1 - \eta\,\varepsilon = \{\partial, D\}$ QED.

<u>Theorem</u> $f \sim g : X \to Y$

$\Rightarrow f_* \simeq g_* : S(X) \to S(Y)$

$(\Rightarrow f_* = g_* : H.(X) \to H.(Y))$

Cor: $X \sim Y \Rightarrow H_*(X) \simeq H_*(Y)$.

Pf: Want $f_* - g_* = \{\partial, D\}$ $\exists D$

D can be constructed as before,

but using $f \sim g$ instead of $\forall f \sim pt : S^n \to X$

§ H_1 = Abelianization of π_1.

<u>Theorem.</u> $H_1(X, \mathbb{Z}) \simeq \pi_1(X) / [\pi_1(X), \pi_1(X)]$

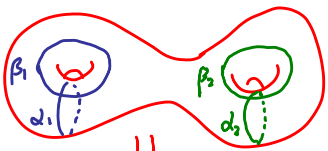
In general, $\pi_q(X) \xrightarrow{\Phi_q} H_q(X)$,

(as $(D^q, S^{q-1}) \to X$ can be treated as cycle,
& homotopy as boundary.)

Not surjective : $\underbrace{\pi_2(\Sigma_g)}_{0} \longrightarrow \underbrace{H_2(\Sigma_g)}_{\mathbb{Z}}$

i.e. Σ_g can be assembled to 2d cycle (w/o bdy),
but cannot use S^2 alone.

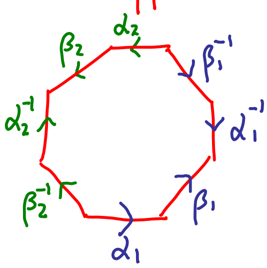
Not injective: $\pi_1(\Sigma_g) \xrightarrow{\Phi_1} H_1(\Sigma_g) \simeq \mathbb{Z}^{2g}$

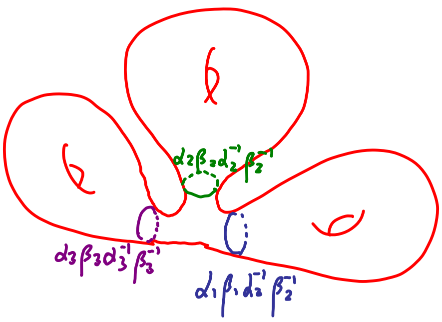


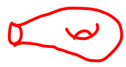
$\langle \alpha_j, \beta_j \rangle_{j=1}^{g}$

$\overrightarrow{\prod_j} \alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1$

($\alpha_j \beta_j \alpha_j^{-1} \beta_j^{-1} = 1$
in multi. notation)



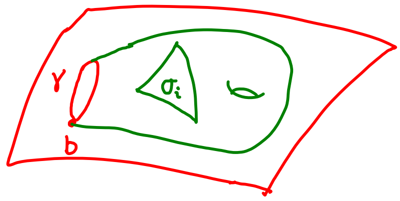


Each $\alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$
bounds  ,
but not disk
$\Rightarrow \Phi_1$ NOT inj.

Thm says this is the only reason $\Phi_1 : \pi_1 \to H_1$ not inj.

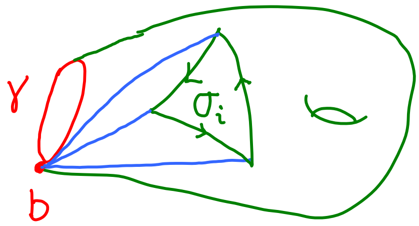
Φ_1 is surjective ✓ ( $\partial\sigma = 0 \Rightarrow$ bdy cancel
$\Rightarrow$ repr. by S^1.)

Pf. of theorem.

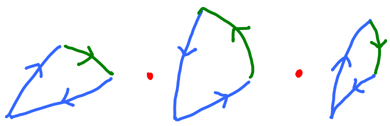


$$\Phi_1(\gamma : S^1 \to X) = 0$$

$$\Rightarrow \quad \gamma = \partial\left(\sum n_i \sigma_i\right)$$



$\forall \sigma_i$

$\beta_i :=$  $\overset{\text{rel.}\,\partial I}{\sim} b$

$(\sim \partial \sigma_i)$

$$\Rightarrow \quad \prod_i [\beta_i]^{n_i} = 1 \quad \in \pi_1(X)$$

LHR \ all repeated  $\rightsquigarrow \gamma$

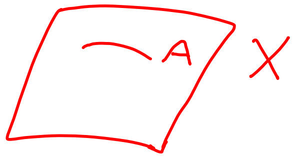
$$\Rightarrow \quad \gamma \left(\underbrace{\prod_i [\beta_i]^{n_i}}_{1}\right) \quad \in [\pi_1, \pi_1]$$

$$\Rightarrow \quad \gamma \in [\pi_1, \pi_1]$$

Indeed $\gamma = \partial f$ $\exists$ $f : \Sigma_g \setminus D^2 \longrightarrow X$.

QED.

§ $A \subset X$



$$\rightsquigarrow \; 0 \to S_\cdot(A) \to S_\cdot(X) \to \frac{S_\cdot(X)}{S_\cdot(A)} \to 0$$

- ∂ decends to quotient

$$\rightsquigarrow \; H_\cdot\left(\frac{S_\cdot(X)}{S_\cdot(A)}, \partial\right) =: H_\cdot(X, A) \quad \text{relative homology}$$

- short exact seq. of <u>complexes</u>.

$\rightsquigarrow$ long exact seq. in homology

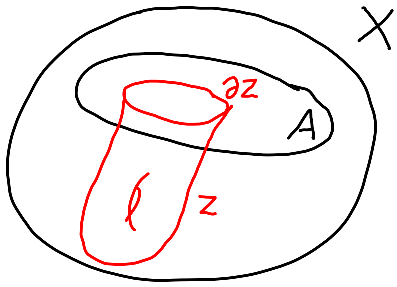
(Standard homological alg. arguments)

$$\cdots \to H_q(A) \to H_q(X) \to H_q(X,A) \xrightarrow{\partial} H_{q-1}(A) \to H_{q-1}(X) \to H_{q-1}(X,A) \to$$

$[\bar{z}] \in H_q(X,A)$ w/ $\bar{z} \in S_q(X)/S_q(A)$ from $z \in S_q(X)$

$\Rightarrow \quad \partial \bar{z} = 0$ in $S_{q-1}(X)/S_{q-1}(A)$

$\Rightarrow \quad \partial z \in S_{q-1}(A)$



$\partial[\bar{z}] := [\partial z] \in H_{q-1}(A)$.

Prop. $A \subset X$ retract $\Rightarrow H_q(X) \cong H_q(A) \oplus H_q(X,A)$

Pf: retract $\iff$ $A \underset{\exists r}{\overset{i}{\rightleftarrows}} X$ s.t. $r \circ i = 1_A$

$$\cdots \to H_q(A) \underset{r_*}{\overset{i_*}{\rightleftarrows}} H_q(X) \to H_q(X,A) \to \cdots\cdots$$

$r_* \circ i_* = 1_{H_\cdot(A)} \xRightarrow[\text{alg.}]{\text{Homological}}$ long ex. seq. split.

Some homological algebras exercises:

- Short exact seq. $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$

$\overset{\Delta}{\Longleftrightarrow}$ $\exists$ $A \xleftarrow{k} B$ s.t. $k \circ i = 1_A$ (Split)

$\Longleftrightarrow$ $\exists$ $B \xleftarrow{l} C$ s.t. $j \circ l = 1_C$

$\Longrightarrow$ $B = A \oplus C$

non-split eg. $0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$

- Direct sum lemma

$$\begin{tikzcd} & A' \arrow[d] \arrow[dr, "\cong"] & \\ A \arrow[r] \arrow[dr, "\cong"'] & B \arrow[r] \arrow[d] & C \\ & C' & \end{tikzcd} \Longrightarrow \begin{array}{c} A \oplus A' \cong B \\ B \cong C \oplus C' \end{array}$$

Both exact at B

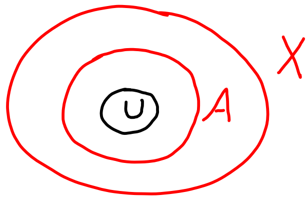
- 5 lemma.

comm. diagram

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "\cong"] & A_2 \arrow[r] \arrow[d, "\cong"] & A_3 \arrow[r] \arrow[d, "\gamma"] & A_4 \arrow[r] \arrow[d, "\cong"] & A_4 \arrow[d, "\cong"] & \leftarrow \text{ex. seq.} \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 & \leftarrow \text{ex. seq.} \end{tikzcd}$$

$\Longrightarrow$ γ $\cong$

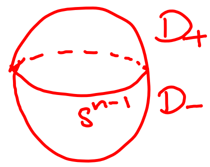
§ Excision



<u>Theorem</u>: $\bar{U} \subset \mathring{A} \Rightarrow$

$$H_q(X \setminus U, A \setminus U) \xrightarrow{\cong} H_q(X, A)$$

Eg. $S^n = D^n_+ \underset{S^{n-1}}{\cup} D^n_-$



(need deform retract to shrink $\mathring{D}_-$ a bit)

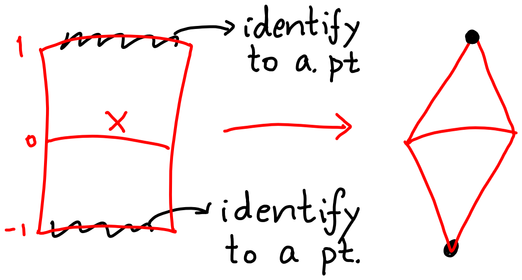
$$\begin{array}{ccc} H_q(\underbrace{S^n \setminus \mathring{D}_-}, \underbrace{D_- \setminus \mathring{D}_-}) & \xrightarrow{\cong} & H_q(S^n, D_-) \\ H_q(\quad D_+ \quad , \quad S^{n-1} \quad) & & \uparrow \cong \text{ if } q>1. \\ \partial \downarrow \cong (\text{if } q>1 \ \because H_{>0}(D_+)=0) & & \\ H_{q-1}(S^{n-1}) & & H_q(S^n) \end{array}$$

$\Rightarrow \quad H_q(S^n) \cong H_{q-1}(S^{n-1})$ when $q-1>0$

$\Rightarrow \quad H_q(S^n) = \begin{cases} \mathbb{Z} & q=n \\ 0 & \text{otherwise.} \end{cases}$ (Easy for $q=1$)

($\xRightarrow{\text{Cor}}$ $S^{n-1} \subset D^n$ not retract $\Longrightarrow$ $D^n \supset f$, $\exists$ fix pt.)

Similarly, $H_q(\underbrace{\Sigma X}_{\text{suspension}}) \simeq H_{q-1}(X)$ when $q-1>0$,

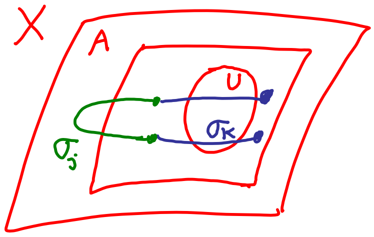


$$= \frac{X \times [-1,1]}{\begin{array}{c} X \times \{-1\} \sim pt \\ X \times \{1\} \sim pt' \end{array}}$$

$X \times [-1]$ $\qquad\qquad$ ΣX

Pf: $H_q(X \setminus U,\ A \setminus U) \longrightarrow H_q(X, A)$

[Onto] $[\sum n_i \sigma_i] \in H_q(X, A)$



Assume all σ_i's are small.

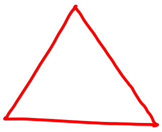
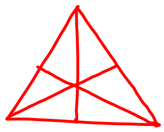
so $\sigma_j \subset X \setminus U$ or $\sigma_k \subset \mathring{A}$

In $H_q(X, A)$, can throw away σ_k's $\subset \mathring{A}$

i.e. $\sum n_j \sigma_j$ w/ $\sigma_j \subset X \setminus U$

hence $[\text{——}\,''\,\text{——}] \in H_q(X \setminus U, A \setminus U)$ $\Rightarrow$ surj.

[1-1] Similar.

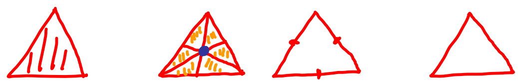
Remain: Replace  by Σ(smaller chains) .

$Sd : S_q(X) \longrightarrow S_q(X)$ Barycentric Subdivision

$Sd\ \delta_q = B_q\ Sd\ (\partial \delta_q)$ B_q = barycenter

$Sd\ \delta_0 = \delta_0$ ·

$Sd\ \delta_2 = B_2\ Sd\ (\partial \delta_2)$



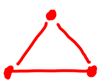
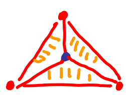
$Sd\ \delta_1 = B_1\ Sd(\partial \delta_1)$

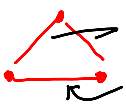
and so on.

Claim. $1_{S(X)} - Sd = \partial T + T \partial$, $\exists\, T : S_q \to S_{q+1}$

$T \delta_q := B_q(\delta_q - Sd\ \delta_q - T \partial \delta_q)$ & $T\delta_0 = 0$.

eg. $T\delta_1 = B_1($  $- \ 0)$

$= B_1$  $=$ 

$\partial T \delta_1 =$ 

$= (1_{S(X)} - Sd)\ \delta_1$ QED.

Mapping cone

$\sim A \subset X \rightsquigarrow \dfrac{S(X)}{S(A)} \rightsquigarrow H_*(X, A)$ & long ex. seq.

$$f : (C_*, \partial) \longrightarrow (C'_*, \partial')$$

$$\Longrightarrow \quad 0 \to C'_* \xrightarrow{i} \underbrace{Cf_*}_{C'_* \oplus C_{*-1}} \xrightarrow{j} \underbrace{C[-1]_*}_{C_{*-1}} \to 0$$

short ex. seq.

$$\partial^{Cf}(c', c) \triangleq (\partial' c' + fc),\ -\partial c)$$

i.e.
$$\partial^{Cf} = \begin{array}{cc} \begin{pmatrix} \partial' & f \\ 0 & -\partial \end{pmatrix} & \begin{array}{c} C' \\ C \end{array} \end{array}$$

$\rightsquigarrow$

$$\cdots\cdots \to H_*(C) \xrightarrow{f} H_*(C') \to H_*(Cf) \to H_{*-1}(C) \to H_{*-1}(C') \to \cdots\cdots$$

long ex. seq.

- $f_* |_{H_\bullet}$ isom $\Longrightarrow H_\bullet(Cf) = 0$

- Conversely, $H_\bullet(Cf) = 0$

$$\Longrightarrow C \underset{\exists g}{\overset{f}{\rightleftarrows}} C' \quad \text{s.t. } f \circ g \simeq 1_{C'} \ \& \ g \circ f \simeq 1_C$$

i.e. chain homotopy.

Remarks of $S^n \subset \mathbb{R}^{n+1} \ni (x_0, x_1, \dots, x_n)$

- $O(n+1) \ni g : S^n \longrightarrow S^n$

$$\Longrightarrow \quad g_*|_{H_n(S^n)} = \pm 1 \quad (= \det g)$$

reason : (i) $O(n+1)$ has 2 connected components $\sim \det g = +1$ or -1.

(ii) For $g =$ reflect^{n} on hyperplane $\left((x_0, \vec{x}) \mapsto (-x_0, \vec{x}) \right)$.
$g_* = -1$ ($n=0$ ✓ ; induct^{n} using $H_\bullet(S^n) \simeq H_{\bullet-1}(S^{n-1})$).

Eg. $(\text{antipodal})_* = (-1)^{n+1}$ on $H_n(S^n)$.

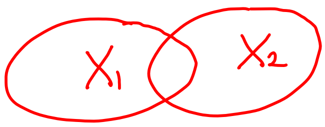
$\exists$ nonvanishing vector field on $S^n \Longleftrightarrow n \in 2\mathbb{Z}+1$

(Pf: [$\Rightarrow$] Move $x \in S^n$ to $-x$ via large circle thru. $v(x)$

[$\Leftarrow$] $S^{2n+1} \subset \mathbb{R}^{2n+2} = \mathbb{C}^{n+1} \rightsquigarrow$ rotate $90°$ in each $\mathbb{C}$)

§ Mayer-Vietoris sequence

$$X = X_1 \underset{A}{\cup} X_2$$



: X

all open sets

Excision for $A \subset X_1$ & $X_2 \subset X$:

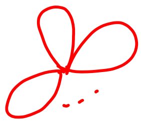
$$\begin{array}{ccccccccccccc}
\cdots \to & H_{q+1}(X_1, A) & \to & H_q A & \to & H_q X_1 & \to & H_q(X_1, A) & \to & H_{q-1} A & \to & H_{q-1} X_1 & \to \cdots \\
& \cong\downarrow & & \downarrow & & \downarrow & & \cong\downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{q+1}(X, X_2) & \to & H_q X_2 & \to & H_q X & \to & H_q(X, X_2) & \to & H_{q-1} X_2 & \to & H_{q-1} X & \to \cdots
\end{array}$$

$\cong \; \because \; X_1 \setminus A = X \setminus X_2$

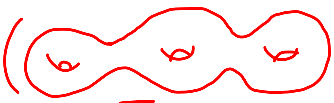
$$\begin{tikzcd}
\cdots \arrow[r] & H_{q+1}(X_1, A) \arrow[r] \arrow[d, "\wr"] & H_q(A) \arrow[r] \arrow[d] & H_q(X_1) \arrow[r] \arrow[d] & H_q(X_1, A) \arrow[r] \arrow[d, "\wr"] & H_{q-1}(A) \arrow[r] \arrow[d] & H_{q-1} X_1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H_{q+1}(X, X_2) \arrow[r] & H_q(X_2) \arrow[r] & H_q(X) \arrow[r] & H_q(X, X_2) \arrow[r] & H_{q-1} X_2 \arrow[r] & H_{q-1} X \arrow[r] & \cdots
\end{tikzcd}$$

$\xRightarrow[\text{chasing}]{\text{Diagram}}$ long exact sequence

$$\cdots \to H_q(A) \longrightarrow H_q(X_1) \oplus H_q(X_2) \longrightarrow H_q(X) \longrightarrow H_{q-1}(A) \to \cdots$$

Eg. $H_*\Big($  $\Big) = \begin{cases} \mathbb{Z}^r & * = 1 \\ 0 & * > 1 \end{cases}$

G_r

$H_*\Big($  $\Big) = \begin{cases} \mathbb{Z}^{2g} & * = 1 \\ \mathbb{Z} & * = 0, 2 \end{cases}$

Σ_g

$(\because\ \Sigma_g \setminus D^2_{\text{small}} \sim G_{2g})$

__Theorem__ (1) $f : S^r \hookrightarrow S^n$ homeo. into

$\Rightarrow\ H^{\#}_q(S^n \setminus f(S^r)) = \begin{cases} \mathbb{Z} & q = n-r-1 \\ 0 & \text{otherwise.} \end{cases}$

In particular, $S^n \setminus f(S^r)$ iff $r = n-1$.

(2) When $r = n-1$,

$S^n \setminus f(S^{n-1}) = K_1 \amalg K_2$ 2 connected components

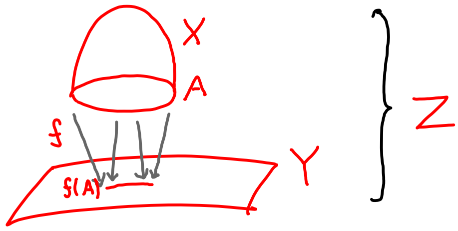
and $\partial K_1 = \partial K_2 = f(S^{n-1})$.

Pf: Skipped.

§ Construct spaces.

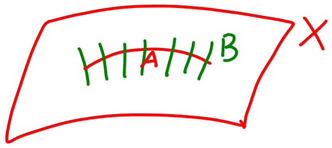
$$X \supset A \xrightarrow{f} Y$$

$$\rightsquigarrow Z \triangleq X \cup_f Y$$



Assume $A \overset{\text{closed}}{\subset} \overset{\text{Hausdorff}}{X}$

- $x \in X \setminus A$ separable from A
- $\exists$ nbd $A \subsetneqq B \subset X$ w/ A as strong deform retract of B, call collar of A



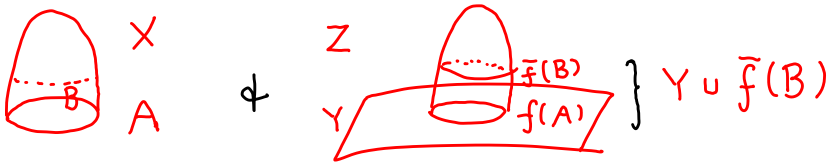
E.g. $A = \partial X$ mfd

- $$X \setminus A = Z \setminus Y$$

<u>Theorem</u> $H_q(X, A) \xrightarrow{\cong} H_q(Z, Y)$

need collar of A to shrink $X \setminus A$ & $Z \setminus Y$ a bit in order to apply excision to

$X \supset B \supset A \qquad Z \supset Y \cup \bar{f}(B) \supset Y$



Attaching n-cell : $\underline{B^n \supset S^{n-1}} \xrightarrow{f} Y \rightsquigarrow Z$

$$\begin{tikzcd}
0 \arrow[r] & H_q(B^n, S^{n-1}) \arrow[r, "\cong"] \arrow[d, "f_*", "\cong"'] & H^{\#}_{q-1}(S^{n-1}) \arrow[r] \arrow[d, "f_*"] & 0 \\
\cdots \to H_q(Y) \to H_q(Z) \arrow[r] & H_q(Z, Y) \arrow[r] & H^{\#}_{q-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

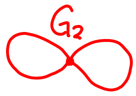
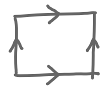
$\Longrightarrow$ all reduced homology

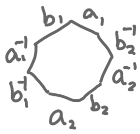
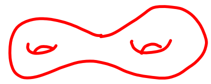
1° $H_q(Z) = H_q(Y)$ if $q \neq n-1, n$

2° $0 \to H_n(Y) \to H_n(Z) \to \mathbb{Z} \xrightarrow{f_*} H_{n-1}(Y) \to H_{n-1}(Z) \to 0$

Spherical complex $\triangleq$ succesive attaching (finite # of) cells starting from points.

Eg $\cdot$ $B^n \supset S^{n-1} \longrightarrow pt \rightsquigarrow Z = S^n$

$\cdot$ $B^2 \supset S^1 \xrightarrow{f}$  $\rightsquigarrow Z = T^2$ $\quad f:$ 

More generally,  etc $\rightsquigarrow Z = \Sigma_g$ 

$$H_q(\Sigma_g) = R, \quad R^{2g}, \quad R$$

$\cdot$ $B^n \supset S^{n-1} \xrightarrow{2:1} \mathbb{RP}^{n-1} \rightsquigarrow Z = \mathbb{RP}^n$

$$H_q(\mathbb{RP}^n) \underset{\text{2-torsion elements}}{\overset{n \text{ even}}{=\!=\!=}} \begin{cases} R & q = 0 \\ R_2 & q = 1, 2, \dots, n \\ 0 & q > n \end{cases} \quad \Bigg| \quad \overset{n \text{ odd}}{=\!=\!=} \begin{cases} R & q = 0 \\ R/2 & q = 1, 2, \dots, n-1 \\ R & q = n \\ 0 & q > n \end{cases}$$

$\cdot$ $B^n_{\mathbb{C}} = B^{2n} \supset S^{2n-1} \xrightarrow[\text{Hopf}]{S^1} \mathbb{CP}^{n-1} \underset{\text{(why?)}}{\rightsquigarrow} Z = \mathbb{CP}^n$

$$H_q(\mathbb{CP}^n) = \begin{cases} \mathbb{Z} & \text{for } q = 0, 2, 4, \dots, 2n, \\ 0 & \text{otherwise.} \end{cases}$$

$\cdot$ $B^{4n} \supset S^{4n-1} \xrightarrow{S^3} \mathbb{HP}^{n-1} \rightsquigarrow Z = \mathbb{HP}^n$

$\cdot$ $B^{16} \supset S^{15} \xrightarrow{S^7} S^8 \rightsquigarrow Z = \mathbb{OP}^2$

Betti number $b_q(X) := \operatorname{rank} H_q(X, \mathbb{Z})$
$$= \dim_{\mathbb{R}} H_q(X, \mathbb{R})$$

Euler characteristic $\chi(X) := \sum_{q=0}^{\infty} (-1)^q\, b_q(X)$

- $A \subset X \rightsquigarrow$ long exact seq. in H_*
$$\Rightarrow \chi(X) = \chi(A) + \chi(X, A)$$

- $B^n \supset S^{n-1} \xrightarrow{f} Y \rightsquigarrow Z$
$$\Rightarrow \chi(Z) = \chi(Y) + (-1)^n$$

Cor. X spherical complex
$\alpha_q :=$ # q-dim. cells used to constr. X
$$\Rightarrow \chi(X) = \sum_{q=0}^{\infty} (-1)^q\, \alpha_q$$

Cor. $\chi(X \times Y) = \chi(X) \times \chi(Y)$

Cor.
$$\chi(X^n \# Y^n) = \begin{cases} \chi(X) + \chi(Y), & n \text{ odd} \\ \chi(X) + \chi(Y) - 2, & n \text{ even} \end{cases}$$

Cor. $E \longrightarrow X$: d-fold cover (say X finite cell cpx)
(say $\exists$ triangulation)
$$\Rightarrow \chi(E) = d \cdot \chi(X)$$

(III) Orientation & Duality on manifolds.

§ Orientation of manifolds

X^n (connected) smooth mfd $\rightsquigarrow$ T_M

$\rightsquigarrow \mathbb{R} \longrightarrow \Lambda^n T_X^* \longrightarrow X$

(loc. trivialization : $f(x)\, dx^1 \wedge \cdots \wedge dx^n$ w/ f nonvanishing)

Orientation

$\Longleftrightarrow$ Global trivialization ν of $\Lambda^n T_X^*$, up to $\times F(x) > 0$

$\Longleftrightarrow$ A conn. component of $\Lambda^n T_X^* \setminus X$

$\Longrightarrow$ $\Lambda^n T_X^* \simeq \underline{\mathbb{R}}$ (i.e. orientable)

- Should not require X smooth (i.e. $\nexists\, T_M$)

1° $\forall x \in X$, $H_n(X, X \setminus x) \simeq R \ni$ $\overset{\text{choose}}{\alpha_x}$ generator

(∵ excision & $H_n(B^n, S^{n-1}) \simeq R$)

2° $\exists$ nbd $U \ni x$, $H_n(X, X \setminus U) \xrightarrow{j^U_{x*}} H_n(X, X \setminus x)$

$\exists\ \alpha_U \longmapsto \alpha_x$

3° $\forall y \in U$, $j^U_{y*}(\alpha_U)$ generates $H_n(X, X \setminus y) \simeq R$

R-Orientation $\overset{\Delta}{\Longleftrightarrow}$ compatible choice of α_U

(equivalent if α_x's are all the same).

$\Rightarrow$ $\overset{\text{(generating)}}{\text{section}}$ of the sheaf $\mathcal{S}$, $\Gamma_c(U, \mathcal{S}) := H_n(X, X \setminus U)$.

So X^n cpt. conn. mfd. $\Rightarrow$ $H_n(X) = \begin{cases} R & \text{if orientable} \\ 0 & \text{otherwise} \end{cases}$ (if R Int. domain)

- $\forall X$, always $\mathbb{Z}_2$-orientable.
- X non-orientable (wrt $R = \mathbb{Z}$) $\Rightarrow \exists$ connected $\tilde{X} \xrightarrow{2:1} X$, $\tilde{X}$ orientable

§ Singular cohomology

$$S^q(X) := S_q(X)^* = \mathrm{Hom}_R(S_q(X), R)$$

$$\delta = \partial^* \downarrow$$

$$S^{q+1}(X) \qquad \delta^2 = 0 \rightsquigarrow H^q(X) := \frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta}\Big|_{S^q(X)}$$

eg. X mfd. $\Omega^q(X) \longrightarrow S^q(X)$ when $R = \mathbb{R}$.

$$\varphi \longmapsto c_\varphi = \left((Y^q \subset X) \mapsto \int_Y \varphi\right)$$

$$d\varphi \longmapsto c_{d\varphi} = \delta(c_\varphi) \quad \text{(Stokes thm)}$$

$$\rightsquigarrow H^q_{dR}(X) \longrightarrow H^q(X)$$

deRham thm says this is isom.

- H^q is Contravariant functor
- long exact seq. for $A \hookrightarrow X \hookrightarrow (X, A)$
- homotopy inv.
- excision
- Mayer-Vietoris seq.
- $\alpha : H^q(X) \longrightarrow H_q(X)^*$

eg. $$\underbrace{H^2(\mathbb{RP}^2, \mathbb{Z})}_{\mathbb{Z}_2} \longrightarrow \underbrace{H_2(\mathbb{RP}^2, \mathbb{Z})^*}_{0}$$

§ Univ. Coeff. theorem

$0 \to \mathrm{Ext}(H_{q-1}(X,\mathbb{Z}),G) \to H^q(X,G) \to \mathrm{Hom}(H_q(X,\mathbb{Z}),G) \to 0$

and $\exists$ non-canonical splitting

- Write $H_q(X,\mathbb{Z}) = \underset{\text{free}}{F_q} \oplus \underset{\text{torsion}}{T_q}$ (eg. $\mathbb{Z}^5 \oplus \mathbb{Z}_2^3$)

  then $H^q \simeq F_q \oplus T_{q-1}$

- $/R$ general, exact seq. $\rightsquigarrow$ spectral seq.

- To define $\mathrm{Ext}(-,-)$, recall resolution for M : R-mod.

$$\cdots\cdots \to C_1 \xrightarrow{\partial} C_0 \xrightarrow{\varepsilon} M \to 0 \quad \text{exact}$$

free (i.e. $R^{\oplus n}$)

- R field $\Rightarrow \exists$ $C_{\geq 1} = 0$, $C_0 = M$.
- $R = \mathbb{Z} \Rightarrow \exists$ $C_{\geq 2} = 0$, $0 \to C_1 \xrightarrow{\partial} C_0 \to M \to 0$.
- $\exists !$ (up to chain homotopy) resol$^{\underline{n}}$.

- $M \rightsquigarrow (C_\bullet, \partial) \overset{N}{\rightsquigarrow} (\mathrm{Hom}(C_\bullet, N), \partial^*)$

$$\mathrm{Ext}^q_R(M,N) := \frac{\mathrm{Ker}\, \partial^*}{\mathrm{Im}\, \partial^*}\Big|_{H^q(\mathrm{Hom}(C_\bullet, N))}$$

- $\mathrm{Ext}^0(-,-) = \mathrm{Hom}(-,-)$

- $\mathrm{Ext}^1(M,N)$ classify extension $\mathcal{E}$ of M by N:

$$0 \to N \to \mathcal{E} \to M \to 0$$

Eg. $\mathrm{Ext}(\mathbb{Z},\mathbb{Z}) = \mathrm{Ext}(\mathbb{Z},\mathbb{Z}_n) = 0$, $q=1$ needed only.

$\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}) = \mathbb{Z}_n$

• $0 \to A \to B \to C \to 0$ induces $(\forall N)$

$$0 \to \mathrm{Hom}_R(C,N) \to \mathrm{Hom}_R(B,N) \to \mathrm{Hom}_R(A,N) \to \mathrm{Ext}^1_R(C,N) \to \mathrm{Ext}^1_R(B,N) \to \mathrm{Ext}^1_R(A,N) \to \mathrm{Ext}^2_R(C,N) \to \mathrm{Ext}^2_R(B,N) \to \mathrm{Ext}^2_R(A,N) \to \cdots$$

(Pf: Diagram chasing

Pf. of Univ. Coeff. thm.

$$0 \to \mathrm{Ext}(H_{n-1}(X,\mathbb{Z}),G) \to \underbrace{H^n(X,G)}_{\frac{\mathrm{Ker}\,\partial^*}{\mathrm{Im}\,\partial^*}\big|_{(S_n)^*}} \to \underbrace{\mathrm{Hom}(H_n(X,\mathbb{Z}),G)}_{(H_n)^*} \to 0 \quad ?$$

/G, /Z, /G, /Z

1° $\quad 0 \to B^*_{n-1} \to \mathrm{Ker}\,\partial^*\big|_{S_n^*} \to H_n^* \to 0$

2° $\quad 0 \to \mathrm{Im}\,\partial^*\big|_{S_n^*} \to B^*_{n-1} \to \mathrm{Ext}(H_{n-1},G) \to 0$

i.e. $H_n^* = \dfrac{\mathrm{Ker}\,\partial^*}{B^*_{n-1}} = \dfrac{\mathrm{Ker}\,\partial^*/\mathrm{Im}\,\partial^*}{B^*_{n-1}/\mathrm{Im}\,\partial^*} = \dfrac{H^n(X,G)}{\mathrm{Ext}(H_{n-1},G)} \Rightarrow$ Done.

As for splitting, choose one for $0 \to Z_n \to S_n \rightleftarrows B_{n-1} \to 0$.

$$\begin{tikzcd}
 & & 0 & S^*_{n+1} & & \\
0 \arrow[r] & H_n^* \arrow[r] & Z_n^* \arrow[r] \arrow[u] & B_n^* \arrow[r] \arrow[u] & \mathrm{Ext}(H_n,G) \arrow[r] & 0 \\
 & 0 & S_n^* \arrow[u] \arrow[ur, bend right, "\partial^*"'] & 0 \arrow[u] & & \\
 & Z^*_{n-1} \arrow[r] \arrow[u] & B^*_{n-1} \arrow[r] \arrow[u] & \mathrm{Ext}(H_{n-1},G) \arrow[r] & 0 & \\
 & S^*_{n-1} \arrow[u] \arrow[uur, bend right, "\partial^*"'] & 0 \arrow[u] & & &
\end{tikzcd}$$

$\mathrm{Ker}\,\partial^*\big|_{S_n^*} = \mathrm{Ker}(S_n^* \to B_n^*)$

$\Downarrow$

H_n^* w/ Kernel $B^*_{n-1} \Rightarrow 1°$. Similar $\Rightarrow 2°$. QED.

Theorem. $[X, S^1] \xrightarrow[\Phi]{\cong} H^1(X, \mathbb{Z})$ if X is finite CW cpx

$f \longmapsto f^*(\mathrm{or}_{S^1})$

In H^*_{dR}, $\Phi(f) = f^*(d\theta)$.

Fact: In general, $H^n(X, G) \simeq [X, K(G,n)]$

w/ $K(G,n)$ s.t. $\pi_n = G$ & $\pi_{\neq n} = 0$ (Eilenberg-MacLane space)

eg. $K(\mathbb{Z}, 1) = S^1$, $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$.

Pf: $\Phi : [X, S^1] \to \mathrm{Hom}(\pi_1(X), \underbrace{\pi_1(S^1)}_{\mathbb{Z}}) \cong \mathrm{Hom}(H_1(X), \mathbb{Z}) \cong H^1(X, \mathbb{Z})$

lifting criterion $\Longrightarrow$ Φ 1-1

X finite CW cpx. $\rightsquigarrow$ 2-Skeleta $X \supset X^2 \simeq (S^1 \vee \cdots \vee S^1) \cup e^2 \cup \cdots \cup e^2$

$\rightsquigarrow$ $1 \to \text{sth} \to \underset{\text{free group}}{F} \to \underbrace{\pi_1(X^2)}_{\pi_1(X)} \to 1$

$\forall\ \varphi : \pi_1(X^2) \to \mathbb{Z} = \pi_1(S^1)$ (as an elt. in $H^1(X, \mathbb{Z})$)

$\rightsquigarrow \exists\ h : X^2 \to S^1$ s.t. $h_* = \varphi$ on π_1 (why?)

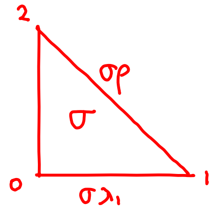
Extend h to whole X ($\because \pi_{>1}(S^1) = 0$) $\Rightarrow$ ✓ QED.

§ Cup product

$$\cup : S^p(X) \otimes S^q(X) \longrightarrow \underbrace{S^{p+q}(X)}_{\mathrm{Hom}(S_{p+q}(X),\,\mathbb{Z})}$$

$$\begin{array}{ccc} & \rotatebox{90}{$\in$} & \\ c & \otimes & d \end{array} \qquad\qquad\qquad \begin{array}{c} \rotatebox{90}{$\in$} \\ \sigma \end{array}$$

$$(c \cup d)(\sigma) \triangleq c(\sigma \underbrace{\lambda_p}_{\text{front } p \text{ dim face}}) \cdot d(\sigma \underbrace{\rho_q}_{\text{back } q \text{ dim face}})$$



$p = q = 1$

- $\delta(c \cup d) = (\delta c) \cup d + (-1)^p c \cup (\delta d)$

$\Rightarrow$ $(H^\bullet(X), \cup)$ graded R-algebra

<u>Theorem</u> $(H^\bullet(X), \cup)$ graded commutative.

i.e. $c \cup d = (-1)^{pq} d \cup c$

Remark: False in $S^\bullet(X)$.

Remark: If X is manifold,

$$\Omega^\bullet(X) \longrightarrow S^\bullet(X), \qquad H^\bullet_{dR}(X) \xrightarrow{\cong} H^\bullet(X, \mathbb{R})$$

$$\wedge \quad \not\longmapsto \quad \cup \qquad\qquad \wedge \quad = \quad \cup$$

Cap product $S_{p+q} \times S^p \xrightarrow{\ \cap\ } S_q$ (adjoint of $\cup$)

$$d(z \cap c) := (c \cup d)(z)$$

$\rightsquigarrow$ $S_\bullet \curvearrowleft S^\bullet$ & $H_\bullet \curvearrowleft H^\bullet$ as right modules.

Pf. of thm.

$$(c \cup d)(\sigma) \triangleq c(\sigma \underbrace{\lambda_p}_{\text{front } p \text{ dim face}}) \cdot d(\sigma \underbrace{\rho_q}_{\text{back } q \text{ dim face}})$$

$\| ??$

$\pm (d \cup c)(\sigma)$ Need to 'swope' front + back faces in any simplex $\sigma \in S_p(X)$

θ : reversing indexes $\overset{\theta}{0 \; 1 \; 2 \; \cdots \; p}$

$\rightsquigarrow \theta : S.(X) \circlearrowleft$ w/ $\theta \partial = \partial \theta$

Claim: $1 - \theta = \partial J + J \partial$ on $S.(X)$ $(\Rightarrow \checkmark)$

Reason: (1) $\forall \sigma$, $C(\sigma) \subseteq S.(X)$ generated by faces of σ (any ordering of vertices),

$\Rightarrow$ $H_{>0}(C(\sigma)) = 0$

(2) General alg. fact: $\phi : S.(X) \to S.(Y)$

$\forall$ simplex σ, $\exists$ acyclic $C(\sigma) \leq S.(Y)$
$\phi(\sigma) \subset C(\sigma) \supset \underset{\forall \text{ face}}{\phi(\sigma^i)}$ } acyclic carrier

$\phi|_{S_0(X)} = 0 \implies \phi \sim 0$ QED.

- Hopf inv. (skip.)

§ Poincaré duality

X^n : R-oriented mfd

IF X compact $\rightsquigarrow$ orientat$^{\underline{n}}$ $\nu \in H_n(X)$

$$D \triangleq \nu \cap (-) : H^q(X) \longrightarrow H_{n-q}(X)$$

<u>Theorem</u> : D $\cong$

Pf. uses 5-lemma for $H^\cdot$ of $U, V, U \cap V \Rightarrow U \cup V$ ✓.
So need a non-compact version as well.

Now X could be non-compact,

Compactly Supported cohomology :

$$H^\bullet_c(X) := \varinjlim_{\text{cpt } K \subset X} H^\bullet(X, X \setminus K)$$

$\left(K_1 \subset K_2 \rightsquigarrow (X, X \setminus K_2) \subset (X, X \setminus K_1) \rightsquigarrow H^\bullet(X, X \setminus K_1) \to H^\bullet(X, X \setminus K_2) \rightsquigarrow \varinjlim_K \right)$

- orientation $\rightsquigarrow$ $\nu_K \in H_n(X, X \setminus K)$

$$\nu_K \cap (-) : H^q(X, X \setminus K) \longrightarrow H_{n-q}(X)$$

(not relative class, $\because$ dual to $H_\bullet(\text{rel}) \times H^\bullet(\text{absolute}) \longrightarrow H_\bullet(\text{rel})$)

Taking $\varinjlim_{\text{cpt } K}$ $\rightsquigarrow$ $D : H^q_c(X) \longrightarrow H_{n-q}(X)$

P.D. : D $\cong$

- $f : X \to Y \not\Rightarrow f_* : H^\cdot_c(X) \to H^\cdot_c(Y)$,

  Instead $f^* : H^\cdot_c(X) \leftarrow H^\cdot_c(Y)$ if f proper.

Pf. of P.D.: 1° $U, V, U\cap V$ ✓ $\xrightarrow[+\,5\text{ lemma}?]{\text{Mayer-Vietoris}}$ $U\cup V$ ✓.

2° For $U = B^n(1)$, enough to take $K = \bar{B}^n(r)$, $r<1$

($\because$ $\forall$ cpt $K' \subset U \Rightarrow K' \subset \bar{B}^n(r)$ $\exists\, r<1$)

$$\nu_{K^n}: H^n(B, B\setminus\bar{B}(r)) \xrightarrow{\cong} H_0(B) \simeq R$$

and all others are zeros. $\Rightarrow$ ✓

3° $X \supset U_{max.}$: max open s.t. P.D. ✓

V any small ball $B^n \subset X$ ($\overset{2°}{\Rightarrow}$ PD ✓)

$U\cap V\ (\subset B^n)$ P.D. ✓ (write $\bigcup_{\text{countable}} B^n(\varepsilon)$ ← good cover (say center $\in \mathbb{Q}^n$))

then 1° $\Rightarrow$ P.D. for $U_{max}\cup V$ $\Rightarrow$ ✓ for X.

- $H^n_c(X^n) \simeq R$
- If X cpt. orientable, $\Rightarrow$

  $b_q(X^n) = b_{n-q}(X)$ and $T_q = T_{n-q-1}$ ↙ torsion of H.
- $H^*(\mathbb{CP}^n, \mathbb{Z}) \overset{\text{alg.}}{=\!=} \mathbb{Z}[\gamma]/\gamma^{n+1}$

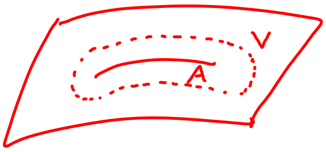
§ Alexander duality

<u>Theorem.</u> Cpt. R-oriented mfd. $X \overset{\text{closed}}{\supset} A \Rightarrow$

$$\cdots\cdots \to H^q_c(X\setminus A) \to H^q(X) \to \check{H}^q(A) \xrightarrow{\delta} H^{q+1}_c(X\setminus A) \to \cdots\cdots$$

Here $\check{H}^\bullet(A) := \varinjlim_{\text{Open } V\supset A} H^\bullet(V)$

($\leq$ ~ reverse inclusion)



$\check{H}^\bullet(A) \longrightarrow H^\bullet(A) \cong$ if A submfd.

(more generally if A has Absolute Nbd. Retract (eg subvariety)

<u>Theorem</u> (Alexander Duality) Same assumption,

$$\Rightarrow \quad D_A : \check{H}^q(A) \xrightarrow{\cong} H_{n-q}(X, X \setminus A)$$

§ Lefschetz Duality

<u>Theorem</u> R-oriented cpt mfd X^n w/ bdy ∂X

$$\begin{tikzcd}
\cdots \arrow[r] & H^{q-1}(X) \arrow[r] \arrow[d, "\nu_n"', "\cong"] & H^{q-1}(\partial X) \arrow[r] \arrow[d, "\partial\nu_n"', "P.D."] & H^q(X, \partial X) \arrow[r] \arrow[d, "\nu_n"', "\cong"] & \cdots \\
\cdots \arrow[r] & H_{n-q+1}(X, \partial X) \arrow[r] & H_{n-q}(\partial X) \arrow[r] & H_{n-q}(X) \arrow[r] & \cdots
\end{tikzcd}$$

(IV) Lefschetz Fixed Point Theorem.

§ Product

Theorem (Künneth formula) R : PID, then

$$H_n(X \times Y) = \bigoplus_p H_p X \otimes H_{n-p} Y + \bigoplus_q \mathrm{Tor}(H_q X, H_{n-q-1} Y)$$

Univ. Coeff. thm.

$$0 \to H_n(X, \mathbb{Z}) \otimes R \to H_n(X, \mathbb{R}) \longrightarrow \mathrm{Tor}(H_{n-1}(X, \mathbb{Z}), R) \to 0$$

§ Thom class & Lef. fix pt. thm.

<u>Theorem.</u> X^n: R-oriented mfd, $\Rightarrow$

$$H^{<n}(X \times X, X \times X \setminus \Delta) = 0$$

$$\exists! \; H^n(X \times X, X \times X \setminus \Delta) \xrightarrow{\phi} \Gamma^* X$$

s.t. $\phi(\beta)(x) = \beta|_{X \times x} \in H^n(X, X \setminus x)$

Thom class μ is s.t. $\phi(\mu) = \nu$ ← ori.

$f: X^n \to Y^m$ both cpt. oriented

$\rightsquigarrow f \times 1 : X \times Y \to Y \times Y$, $\mu_Y' \in H^m(Y \times Y)$ image of Thom class of Y

$\rightsquigarrow \mu_f := (f \times 1)^* \mu_Y' \in H^m(X \times Y)$

- $f^*(-) = \pm \mu_f / \nu_Y \cap (-) : H^\cdot(Y) \to H^\cdot(X)$

Now $f: X \circlearrowleft$ (i.e. $X = Y$)

- $\mu_f \neq 0 \Rightarrow f(x) = x \;\; \exists x$
- $L_f := \Delta^* \mu_f \in H^n(X)$
- Lefschetz number $\Lambda_f := L_f(\nu_X)$ ($\neq 0 \Rightarrow \exists$ fix pt.)

Theorem $\Lambda_f = \sum_q (-1)^q \, \mathrm{Tr}(f^* : H^q(X) \circlearrowleft)$